A Survey on Eigenvalues for Nonnegative Tensors

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A Survey on the Spectral Theory of Nonnegative Tensors

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Contents

0. Why we study eigenvalues for nonnegative tensors?

In this talk, a tensor(supermatrix) is understood as an array of data:

$$\mathbf{A} = (a_{i_1 i_2 \cdots i_m})\ 1 \leq i_1 i_2 \cdots i_m \leq n.$$

where $a_{i_1 i_2 \cdots i_m}$ are numbers, real or complex.
The n^m numbers arranged in this way is called an m order, n dimensional tensor.
It is an extension of matrix, a matrix is a 2 order tensor.

In 2005, Qi and Lim independently proposed to find $(\lambda_0, x_0) \in \mathbb{R}^1 \times (\mathbb{R}^n \backslash \{0\})$ satisfying

◦ (The $H-$ eigenvalue problem)

$$\Sigma^n_{i_2,\cdots,i_m=1} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}, \quad i = 1, 2, \cdots, n,$$

◦ (The $Z-$ eigenvalue problem)

$$\begin{cases} \Sigma^n_{i_2,\cdots,i_m=1} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i, \quad i = 1, 2, \cdots, m, \\ \Sigma^n_{i=1} x_i^2 = 1 \end{cases}$$

◦ According to Qi, the LHS of these equations is denoted by $\mathbf{A}x^{m-1} \in \mathbb{R}^n$.

◦ Eigenvalues only depends on the n homogeneous polynomials: $(\mathbf{A}x^{m-1})_i, \quad i = 1, \cdots n.$ $\Rightarrow$ May assume the $(m-1)$-order n-dimensional tensor $(a_{ii_2\cdots i_m})$ is symmetric, $\forall\, i$.

(1). Best rank-1 approximation

$\mathbf{B} \in \mathbb{R}^{[m,n]}$ is called Rank-1, if $\exists\,(\lambda, u) \in \mathbb{R}^1 \times S^{n-1}$, such that $\mathbf{B} = \lambda u^{\otimes m}$, where $u = (u_1, \cdots u_n)$, and

$$u^{\otimes m} = \Sigma^n_{i_1,\cdots i_m} u_{i_1} \cdots u_{i_m}.$$

$\mathbf{A} \in \mathbb{R}^{[m,n]}$ is called symmetric [Qia]if

$$a_{i_1\cdots i_m} = a_{\sigma(i_1\cdots i_m)} \quad \text{for all } \sigma \in \mathrm{S}_m,$$

where S_m denotes the permutation group of m indices.
Given a symmetric tensor $\mathbf{A}$ we want to find a rank-1 tensor $\mathbf{B} = \lambda u^{\otimes m}$ such that

$$\|\mathbf{A} - \mathbf{B}\|^2_F = \min_{(\lambda,v)\in\mathbb{R}^1\times S^{n-1}} \Sigma^n_{i_1\cdots i_m} |a_{i_1\cdots i_m} - \lambda v_{i_1} \cdots v_{i_m}|^2.$$

$\Leftrightarrow$

$$\begin{cases} \lambda = \mathbf{A}u^m = \sum_{i_1,\cdots,i_m=1}^n a_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m}, \\ \mathbf{A}u^m = \max_{v\in S^{n-1}} \mathbf{A}v^m \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \mathbf{A}u^{m-1} = \lambda u, \\ \sum_{i=1}^n u_i^2 = 1, \\ \lambda = \mathbf{A}u^m \end{cases}$$

$\Rightarrow u$ is a *Z*-eigenvalue of $\mathbf{A}$.

L. De Lathauwer, B. De Moor and J. Vandewalle [LMV] (2000), E. Kofidis and P. Regalia [KR] (2002), Qi [Qia](2005).

(2). Spectrum for Hypergraph Let $G = (V, E)$ be a graph .
(Vertex set) $V = \{1, 2, \cdots, n\}$,
(Edge set) E is a set of paths (or edges). A path connecting vertex i and vertex j is denoted by $e = e_{ij}$.
Adjacency matrix $A = (a_{ij})$

$$a_{ij} = \begin{cases} 1 & \text{if } \exists\, e_{ij} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The spectrum of a graph provides many important information of the graph. Graph spectral theory becomes a part of graph theory.
$H = (V, E)$ is said to be a hypergraph , if each edge $e \in E$ is a subset of V . It is called m-uniform for an integer $m \geq 2$, if for all $e \in E(H)$, $|e| = m$. (Duchet)
The counterpart of the adjacency matrix now is the adjacency tensor $\mathbf{A}_H = (a_{i_1,\cdots,i_m})$ defined by

$$a_{i_1,\cdots,i_m} = \begin{cases} 1, & \text{if } i_1, \cdots i_m \in E \\ 0, & \quad \text{otherwise} \end{cases}$$

Lim applied $H-$eigenvalues to study hypergraphs.
Recently,

◦ Cooper and Dutle [CD] studied the largest modulus of a m-grpah.

◦ By using the spectral radius of the hypergraph H, Bulo and Pelillo [BP, BP1] obtained new upper and lower bounds for the clique number $\omega(G)$ of a undirected graph G.

see also, Hu and Qi [HQ-2], J. Xie, A. Chang [XC], and K. Pearson, and T. Zhang [PT].

(3). High order Markov Chain

In analyzing data sequences in different areas, W. Ching and M. Ng [CN] and Ng et al. [LNY][LN],[N] employed high order Markov chain models as a new mathematical tool. This leads to eigenvalue problems for nonnegative tensors.

A higher-order Markov chain is an extension of the finite Markov chain, in which the stochastic process $X_0, X_1, \cdots$ with values in $\{1, 2, \cdots, n\}$ has the transition probabilities:

$$0 \leq p_{i_1 i_2 \cdots i_m} = \mathrm{Prob}(X_N = i_1 \mid X_{N-1} = i_2, \ldots, X_{N-m+1} = i_m) \leq 1$$

where

$$\sum_{i_1=1}^{n} p_{i_1, i_2, \cdots, i_m} = 1, \quad 1 \leq i_2, ..., i_m \leq n, \quad (Sto).$$

A tensor $\mathbf{P} \in \mathbb{R}_+^{[m,n]}$:

$$\mathbf{P} = (p_{i_1 i_2 \ldots i_m}), \quad 1 \leq i_1, i_2, ..., i_m \leq n,$$

satisfying (Sto), is called a transition probability tensor.

Let the probability distribution at time N be $\xi^{(N)} \in \Delta_n$, where

$$\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, \sum_{j=1}^{n} x_j = 1\}.$$

Then we have

$$\xi^{(N+m)} = \left(\sum_{i_2 \cdots i_m = 1}^{n} p_{i,i_2 \cdots i_m} \xi_{i_2}^{(N+m-1)} \cdots \xi_{i_m}^{(N)} \right)_{i=1}^{n} \in \Delta_n, \quad N = 1, 2, \cdots .$$

If

$$\lim_{N \to \infty} \xi^{(N)} = \xi$$

exists, then ξ satisfies

$$\begin{cases} \mathbf{P} x^{m-1} = x, \\ x \in \Delta_n. \end{cases}$$

ξ is called the stationary probability distribution of the higher-order Markov chain.

This is a new kind of eigenvalue problem:

$$\begin{cases} \mathbf{P}x^{m-1} = \lambda x, \\ x_i \geq 0, \;\; i = 1, \cdots, n, \\ \Sigma_{i=1}^n x_i = 1. \end{cases}$$

It is called a Z_1 eigenvalue problem.

$\Rightarrow \quad \lambda = 1.$

Although the Z_1 eigenvalue problem is different from the Z-eigenvalue problem for $\mathbf{P}$,
they share the same eigenvectors (with a positive constant multiplier),
but correspond to different eigenvalues.

(4) Other applications.

◦ Diffusion kurtosis tensors, (Qi, Wang and Wu [QWW]; Qi, Yu and Wu [QYW]; Hu, Huang, Ni and Qi [HHNQ] and Qi, Yu and Xu [QYX] 2007-2009)

◦ Multi-relation data mining, (X. Li, M. Ng, Y. Ye [LNY] 2011)

◦ Illumination Detection of an Image, (Zhang, Zhou, Peng 2011)

◦ The quantum entanglement problem is related to $Z-$eigenvalue problem, (see S. Hu, L. Qi, and G. Zhang [HQZ] 2012).

◦ High order Taylor expansions

◦ High order moments of statistical quantities

1. P-F Theorem for nonnegative matrices

Theorem

(Weak Form) *If A is a nonnegative square matrix, then*

1. *$r(A)$, the spectral radius of A, is an eigenvalue.*
2. *There exists a nonnegative vector $x_0 \gneqq 0$ such that*

$$Ax_0 = r(A)x_0.$$

Definition

A square matrix A is said to be reducible if it can be placed into block upper-triangular form by simultaneous row/column permutations. A square matrix that is not reducible is said to be irreducible.

Theorem

(Strong Form) *If A is an irreducible nonnegative square matrix, then*

1. $r(A) > 0$ *is an eigenvalue.*
2. *There exists a positive vector* $x_0 > 0$, *i.e. all components of* x_0 *are positive, such that* $Ax_0 = r(A)x_0$.
3. *(Uniqueness) If λ is an eigenvalue with a nonnegative eigenvector, then* $\lambda = r(A)$.
4. $r(A)$ *is a simple eigenvalue of A.*
5. *If λ is an eigenvalue of A, then* $|\lambda| \leq r(A)$.

Concerning the distribution of eigenvalues on the spectral circle $\{\lambda \in C \mid |\lambda| = r(A)\}$,

Theorem

Let A be an irreducible nonnegative matrix. If A has k distinct eigenvalues of modulus $r(A)$, then the eigenvalues are $r(A)e^{i2\pi j/k}$, *where* $j = 0, 1, \cdots, k-1$.

We call the number k the cyclic index of A.

Definition

An irreducible nonnegative matrix A is said to be primitive if the only nonempty subset of the boundary of the positive cone P in R^n, which is invariant under the action of A is $\{0\}$.

In particular, if A is a positive matrix, then A is primitive.

Theorem

A is a primitive matrix if and only if A has cyclic index 1.

Corollary

If A is a positive matrix, and $\lambda \neq r(A)$ is an eigenvalue of A then $|\lambda| < r(A)$.

There is also a minimax characterization of the spectral radius for irreducible nonnegative matrices due to Collatz.

Theorem

(Collatz) *Assume A is an irreducible nonnegative $n \times n$ matrix, then*

$$\min_{x \in P^\circ} \max_{\{i | x_i > 0\}} \frac{(Ax)_i}{x_i} = r(A) = \max_{x \in P^\circ} \min_{\{i | x_i > 0\}} \frac{(Ax)_i}{x_i},$$

Power method in computing $r(A)$

Let $A \geq 0$ be an $n \times n$ irreducible matrix. For any initial value $y^{(0)} \in P^{\circ}$, the interior of P, let $x^1 = \|y^0\|^{-1} y^0$. We compute iteratively

$$y^{(r)} = Ax^{(r)}, \; x^{(r)} = \|y^{(r-1)}\|^{-1} y^{(r-1)} \quad r \geq 1.$$

Compute

$$\overline{\lambda}_r = \max_{1 \leq i \leq n} \frac{y_i^{(r)}}{x_i^{(r)}}, \; \underline{\lambda}_r = \min_{1 \leq i \leq n} \frac{y_i^{(r)}}{x_i^{(r)}};$$

We have,

$$\underline{\lambda}_0 \leq \underline{\lambda}_1 \leq \cdots \leq r(A) \leq \cdots \leq \overline{\lambda}_1 \leq \overline{\lambda}_0.$$

Conclusion:

Theorem

If A is primitive, then both the sequences $(x^{(r)}, \underline{\lambda}_r)$ and $(x^{(r)}, \overline{\lambda}_r)$, produced by the power method, converge to $(x_0, r(A))$, where x_0 is the positive eigenvector corresponding to the eigenvalue $r(A)$.

Theorem

(Qi [Qia] 2005) If $\mathbf{A} \in \mathbb{R}^{[m,n]}$ *is symmetric, then*

1. *A number* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $\mathbf{A}$ $\Leftrightarrow$ *a root of the characteristic polynomial* $\phi(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, *where* $\mathbf{I} = (\delta_{i_1\cdots i_m})$ *denotes the identity tensor.*
2. *The number of eigenvalues of* $\mathbf{A}$ *is* $d = n(m-1)^{n-1}$. *Their product is equal to* $\det(\mathbf{A})$, *the resultant of* $\mathbf{A}x^{m-1} = 0$.
3. *The sum of all the eigenvalues of* A *is* $(m-1)^{n-1}\mathrm{tr}(\mathbf{A})$, *where tr*$(\mathbf{A})$ *denotes the sum of the diagonal elements of* $\mathbf{A}$.
4. *If* m *is even, then* $\mathbf{A}$ *always has H-eigenvalues.* $\mathbf{A}$ *is positive definite (positive semidefinite)* $\Leftrightarrow$ *all of its H-eigenvalues are positive (nonnegative).*
5. *The eigenvalues of* $\mathbf{A}$ *lie in the following* n *disks:*

$$|\lambda - a_{ii\cdots i}| \leq \sum_{i_2,\cdots,i_m \neq i} |a_{ii_2\cdots i_m}|, \ \forall\, 1 \leq i \leq n.$$

In fact, the symmetric assumption on $\mathbf{A}$ in [Qia] is superfluous. The determinant can be defined as the resultant of these polynomials:

$$\det(\mathbf{A}) = res((\mathbf{A}x^{m-1})_1, \cdots (\mathbf{A}x^{m-1})_n),$$

then the characteristic polynomial becomes

$$\phi(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}).$$

Canny [Can] defined the generalized characteristic polynomial (GCP), $C(\lambda)$, of a system of homogeneous polynomials $f_1, \ldots, f_n$ in the variables $x_1, \ldots, x_n$ to be the resultant of $\{f_1 - \lambda x_1^{d_1}, \ldots, f_n - \lambda x_n^{d_n}\}$, where each f_i has total homogeneous degree d_i.

$$\sigma(\mathbf{A}) = \{\lambda \,|\, \lambda \text{ is an eigenvalue of } \mathbf{A}\}$$

is called the spectrum of $\mathbf{A}$.

$\Rightarrow \sigma(\mathbf{A}) \neq \emptyset$ is a finite set.

$$\rho(\mathbf{A}) = \max\{|\lambda| \,|\, \lambda \in \sigma(A)\}.$$

is called the spectral radius[YYa] .

Lim [Lima] first proposed to extend the Perron-Frobenius Theorems to nonnegative tensors in this setting.

Theorem

(Chang, Pearson, and Zhang 2008 [CPZPF]) If $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$*, then there exist* $\lambda_0 \geq 0$ *and a nonnegative vector* $x_0 \neq 0$ *such that*

$$\mathbf{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}. \tag{H}$$

The proof is based on Brouwer fixed point theorem. Lim [Lima] also extended the notion of irreducibility to higher order tensors.

Definition

A tensor $\mathbf{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1 \cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathbf{A}$ is not reducible, then we call $\mathbf{A}$ irreducible.

Let $P^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \forall\, i\}$ be the positive cone in $\mathbb{R}^n$.

Theorem

(Chang, Pearson, and Zhang 2008 [CPZPF]) If $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ is irreducible, then the eigenpair (λ_0, x_0) in (H) satisfies:

1. *(Positivity)* $\lambda_0 > 0$ *and* $x_0 > 0$, *i.e. all components of* x_0 *are positive.*
2. *(Uniqueness)* *If* λ *is an eigenvalue with nonnegative eigenvector, then* $\lambda = \lambda_0$.
3. *(Positively simple)* *the nonnegative eigenvector is unique up to a multiplicative constant.*
4. *(Largest modulus)* *If* λ *is an eigenvalue of* $\mathbf{A}$, *then* $|\lambda| \leq \lambda_0$.

Corollary

(Yang and Yang [YYa]) $\forall\ \mathbf{A} \in \mathbb{R}_+^{[m,n]}$, $\rho(\mathbf{A})$ *is an eigenvalue of* $\mathbf{A}$.

Theorem

(Yang and Yang [YYa]) Let $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ *be irreducible. If* $\mathbf{A}$ *has* k *distinct eigenvalues of modulus* $\rho(A)$*, then the eigenvalues are* $\rho(A)e^{i2\pi j/k}$*, where* $j = 0, 1, \cdots, k-1$.

Theorem

(Chang Pearson and Zhang [CPZPFco]) Assume $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ *is irreducible, then*

$$\min_{x \in (P^n)^\circ} \max_{\{i | x_i > 0\}} \frac{(\mathbf{A}x^{m-1})_i}{x_i^{m-1}} = \rho(\mathbf{A}) = \max_{x \in (P^n)^\circ} \min_{\{i | x_i > 0\}} \frac{(\mathbf{A}x^{m-1})_i}{x_i^{m-1}}.$$

Inspired by Theorem [CPZPFco] and Power method for matrices, Ng, Qi, and Zhou [NQZ] proposed the following algorithm for calculating the spectral radius:

1. Choose $x^0 \in (P^n)^\circ$. Let $y^0 = \mathbf{A}(x^{(0)})^{m-1}$ and set $k := 0$.
2. Compute

$$x^{(k+1)} = \frac{(y^{(k)})^{[\frac{1}{m-1}]}}{\|(y^{(k)})^{[\frac{1}{m-1}]}\|},$$

$$y^{(k+1)} = \mathbf{A}(x^{(k+1)})^{m-1},$$

$$\underline{\lambda}_{k+1} = \min_{1\le i\le n} \frac{(y^{(k+1)})_i}{(x_i^{(k+1)})^{m-1}},$$

$$\overline{\lambda}_{k+1} = \max_{1\le i\le n} \frac{(y^{(k+1)})_i}{(x_i^{(k+1)})^{m-1}}.$$

If the iteration does not terminate in finite time, are the sequences $\{\underline{\lambda}_k, x^{(k)}\}$ $\{\overline{\lambda}_k, x^{(k)}\}$ convergent?

By defining the nonlinear map on P^n associated with the tensor $\mathbf{A}$, one defines a map:

$$\mathbf{T}_{\mathbf{A}} x := (\, A x^{m-1})^{[\frac{1}{m-1}]},$$

Chang Pearson and Zhang [CPZP] enabled the composition of the tensor $\mathbf{A}$ with itself and extended the definition of primitivity to tensors.

Definition

An irreducible nonnegative tensor $\mathbf{A}$ is said to be primitive if the only nonempty subset of the boundary of the positive cone P^n, which is invariant under $\mathbf{T}_{\mathbf{A}}$ is $\{0\}$.

Theorem

[CPZP] Let $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$, then the following statements are equivalent:

1. $\mathbf{A}$ *is primitive.*
2. $\exists\, r \in \mathbf{N}$ *such that* $\mathbf{T}_{\mathbf{A}}^r(P^n \backslash \{0\}) \subset (P^n)^\circ$, *i.e.,* $\mathbf{T}_{\mathbf{A}}^r$ *is strongly positive.*
3. $\exists\, r \in \mathbf{N}$ *such that* $\mathbf{T}_{\mathbf{A}}^r$ *is strictly increasing.*

Theorem

[CPZP] If $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ *is primitive, then its cyclic index is* 1.

Theorem

[CPZP] Let $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ *be irreducible. Both the sequences* $\{\underline{\lambda}_k\}$ *and* $\{\overline{\lambda}_k\}$ *converge to* $\rho(\mathbf{A})$ *for an arbitrary initial value* $x^0 \in P^n \backslash \{0\}$ *if and only if* $\mathbf{A}$ *is primitive.*

Corollary

[CPZP] Let $\mathbf{A} \geq 0$ *be irreducible. Then* $\mathbf{A} + \alpha\mathbf{I}$ *is primitive, where* $\mathbf{I}$ *is the identity tensor and* $\alpha > 0$*.*

Corollary

[CPZP] If $\mathbf{A} \geq 0$ *is essentially positive, then* $\mathbf{A}$ *is primitive.*

These corollaries imply the convergence results in Qi and Zhang [QZ] and Yang, Yang and Li [YYL].

In particular, for any irreducible $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$, one may use the iteration proposed by Ng Qi and Zhou to the modified tensor $\mathbf{A} + \alpha\mathbf{I}$, (which is primitive). And subtract α after finding the largest eigenvalue of $\mathbf{A} + \alpha\mathbf{I}$.

Recently, Friedland, Gaubert and Han [FGH] introduced the notion of weakly irreducible nonnegative tensors. Given $\mathbf{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$, it is associated to a directed graph $G(\mathbf{A}) = (V, E(\mathbf{A}))$, where $V = \{1, 2, \cdots, n\}$ and a directed edge $(i, j) \in E(\mathbf{A})$ if there exists indices $\{i_2, \cdots, i_m\}$ such that $j \in \{i_2, \cdots, i_m\}$ and $a_{ii_2 \cdots i_m} > 0$, i.e.,

$$\Sigma_{j \in \{i_2, \cdots, i_m\}} a_{ii_2 \cdots i_m} > 0.$$

Definition

A nonnegative tensor $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ is called weakly irreducible if the associate directed graph $G(\mathbf{A})$ is strongly connected.

It is equivalent Hu ([Hu]) to say: $\Leftrightarrow$ the matrix $M(\mathbf{A}) = (m_{ij})$ is irreducible, where

$$m_{ij} = \Sigma_{j \in \{i_2, \cdots, i_m\}} a_{ii_2 \cdots i_m}.$$

irreducible $\Rightarrow$ weakly irreducible

Example

[YYc] Let $\mathbf{A} \in \mathbb{R}_{+}^{[4,3]}$ be given by

$$a_{1111} = a_{1123} = a_{2223} = a_{3113} = 1 \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

This is a reducible, weakly irreducible tensor.

Meanwhile, Friedland, Gaubert and Han [FGH] discovered that a series of results obtained by Nussbaum [Nusa, Nusb], Burbanks, Nussbaum, and Sparrow [BNS], and Gaubert, Gunawardena [GG] etc. on order preserving mappings as well as on positively 1-homogeneous monotone functions can be applied to the nonnegative tensors setting. Applying these results, they reproved Theorem P-R Theorem (strong form) under the weakly irreducible condition:

Theorem

[FGH] Assume that $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ is weakly irreducible. Then there exists a unique positive H-eigenvector with positive eigenvalue.

A comparison

1. (irreducible) existence in $(P^n)^\circ$, uniqueness in $P^n \backslash \{0\}$.
2. (weakly) existence in $(P^n)^\circ$, uniqueness in $(P^n)^\circ$.

Comparing results for tensors and the classical for nonnegative matrices.

The following THREE PROPERTIES are the same:

(Positivity) for eigenpair $(\lambda_0, x_0) \in \mathbb{R}^1_+ \times (P^n)^\circ$.

(Uniqueness) for eigenvalue with nonnegative eigenvector.

(Largest modulus) $|\lambda| \le \lambda_0 \ \forall \lambda \in \sigma(\mathbf{A})$.

The difference between nonnegative matrices and tensors:

$$\text{irreducible matrices} \to \begin{cases} \text{weakly irreducible tensors} \\ \text{irreducible tensors.} \end{cases}$$

Geometrically simple $\leftarrow | \rightarrow$ Positively simple

Definition

([CPZPF]) Let $\lambda \in \sigma(\mathbf{A})$, λ is called real geometric multiplicity q, if the maximum number of linearly independent real eigenvectors corresponding to λ equals q. If $q = 1$, then λ is called real geometrically simple.

Example

([CPZPF]) Let $\mathbf{A} = (a_{ijk}) \in \mathbb{R}_+^{[3,2]}$ be such that $a_{111} = a_{222} = 1$, $a_{122} = a_{211} = \epsilon$ for $0 < \epsilon < 1$, and $a_{ijk} = 0$ for other (ijk). Then the H eigenvalue problem reads as

$$\begin{cases} x_1^2 + \epsilon x_2^2 = \lambda x_1^2 \\ \epsilon x_1^2 + x_2^2 = \lambda x_2^2. \end{cases}$$

We have $\lambda_0 = 1 + \varepsilon$, with eigenvectors: $u_1 = (1, 1)$ and $u_2 = (1, -1)$.
$\Rightarrow$ the real geometric multiplicity of $\lambda_0 = 1 + \varepsilon$ is 2.

When m is even,

$$\mathbf{T}_A(x) = (\mathbf{A}x^{m-1})^{\frac{1}{m-1}},$$

is well defined on $\mathbb{R}^n$, 1-homogeneous, and maps P^n to P^n. In this case,

$$\mathbf{A}x^{m-1} = \lambda x^{m-1} \;\Leftrightarrow\; \mathbf{T}_A x = \lambda^{\frac{1}{m-1}} x, \quad \forall\, x \in \mathbb{R}^n.$$

It follows directly

Corollary

([YYb]) Let $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ be irreducible and m is even. Then $\rho(\mathbf{A})$ is real geometrically simple.

$\Rightarrow$ Yang-Yang [YYa] and Pearson [KJP] (for essential positive tensors i.e., $\mathbf{T}_A$ is strongly positive) on the geometric simplicity of the largest eigenvalue.

The example [YYc], which is reducible but not weak irreducibility yields two positive eigenvectors:
$x_1 = (-0.410215, 0.231207, 0.33885)$, and
$x_2 = (5.03736, 2.83918, 4.16102)$, corresponding to $\rho(\mathbf{A}) \approx 1.46557$.
This means: $\rho(\mathbf{A})$ of a nonnegative even order weakly irreducible tensor is not real geometrically simple.

Various extensions

◦ Zhang and Qi [QZ] studied the weakly positive tensors.

◦ Hu, Huang, and Qi [HHQ] studied the strictly nonnegative tensors.

◦ Further developments with regards to the algorithms can be found in [LZI, YYL, ZCQ, ZQX, ZQW].

◦ Extensions to rectangular nonnegative tensors, essentially nonnegative tensors and M-tensors can be found in [CQZ, CZ, KJP, YY2, Zh, ZQL, ZQZ, ZCQ].

3. The Z-spectral theory for nonnegative tensors

Definition

([Qib]) Let $\mathbf{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector of $\mathbf{A}$ if they satisfy the equation

$$\begin{cases} \mathbf{A}x^{m-1} = \lambda x, \\ x^T x = 1 \end{cases}$$

We call (λ, x) a Z-eigenpair if they are both real.

∘ Z eigenpairs are orthogonally invariant! [Qi]

The E-characteristic polynomial of **A** is defined as ([Qi])

$$\psi_{\mathbf{A}}(\lambda) = res_x(\mathbf{A}x^{m-1} - \lambda(x^T x)^{\frac{m-2}{2}} x), \;\; m \text{ is even,}$$
$$\psi_{\mathbf{A}}(\lambda) = res_{x,x_0}(\mathbf{A}x^{m-1} - \lambda x_0^{m-2} x, x^\top x - x_0^2), \;\; m \text{ is odd.}$$

We say that **A** is regular if the following system has no nonzero complex solutions:

$$\begin{cases} \mathbf{A}x^{m-1} = 0, \\ x^T x = 0 \end{cases}$$

Theorem

Qi [Qia],[Qib]

1. *If* $\mathbf{A}$ *is regular, E-eigenvalue of* $\mathbf{A}$ $\Leftrightarrow$ *root of* $\psi_{\mathbf{A}}$.
2. *If* $\mathbf{A}$ *is symmetric, then the Z-eigenvalues always exist. An even order symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive.*
3. *The E-characteristic polynomial is orthogonal invariant.*
4. *If* λ *is the Z-eigenvalue of* $\mathbf{A}$ *with the largest absolute value and* x *is a Z-eigenvector associated with it, then* λx^m *is the best rank-one approximation of* $\mathbf{A}$*, i.e.,*

$$\begin{aligned}||\mathbf{A} - \lambda x^m||_F &= \sqrt{||\mathbf{A}||_F^2 - \lambda^2}\\ &= \min\{||\mathbf{A} - \alpha u^m||_F : \alpha \in \mathbb{R}, u \in \mathbb{R}^n, ||u||_2 = 1\},\end{aligned}$$

where $||\cdot||_F$ *is the Frobenius norm.*

In contrast to $\sigma(\mathbf{A})$, the E spectrum may be unbounded.

Example

Let $\mathbf{A} = (a_{ijk}) \in \mathbb{C}^{[3,2]}$, where

$$a_{111} = a_{221} = 1, \quad a_{112} = a_{222} = i, \quad a_{ijk} = 0 \text{ otherwise.}$$

We solve the system

$$\begin{cases} x_1^2 + ix_1x_2 = \lambda x_1, \\ x_1x_2 + ix_2^2 = \lambda x_2 \end{cases}$$

It is easily seen that all $\lambda \neq 0$ are E-eigenvalues of $\mathbf{A}$.

On the existence of $Z-$ eigenvalue for nonnegative tensors

Theorem

([CPZ1]) If $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$*, then there exists a Z-eigenpair* $(\lambda_0, x_0) \in \mathbb{R}_+^1 \times P^n$*. If further, If* $\mathbf{A}$ *is irreducible, then the pair* $(\lambda_0, x_0) \in (\mathbb{R}_+^1)^\circ \times (P^n)^\circ$.

Comparing with H eigenvalues for nonnegative tensors, the existence of positive eigenpair is the same. But there is NO UNIQUENESS!

Example

Let $\mathbf{A} \in \mathbb{R}_+^{[4,2]}$ be defined by

$$a_{1111} = a_{2222} = \frac{4}{\sqrt{3}}, \quad a_{1112} = a_{1121} = a_{1211} = a_{2111} = 1,$$
$$a_{1222} = a_{2122} = a_{2212} = a_{2221} = 1, \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

It is irreducible and has two positive Z-eigenvalues:
$(\lambda_0, x_0) = (2 + \frac{2}{\sqrt{3}}, (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}))$.
$\lambda_1 = \frac{11}{2\sqrt{3}}$ with Z-eigenvectors: $x_1 = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, and $x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.
∘ the eigenvalue λ_1 is not positively simple!
Similar to H-eigenvalues, we may define the Z-spectrum of $\mathbf{A}$ as follows.

Definition

Let $\mathbf{A} \in R^{[m,n]}$. We define the Z-spectrum of $\mathbf{A}$,

$$\mathbf{Z}(\mathbf{A}) = \{\lambda \,|\, Z - \text{eigenvalues of} \mathbf{A}\}.$$

Assume $\mathbf{Z}(\mathbf{A}) \neq \emptyset$, define the Z-spectral radius of $\mathbf{A}$,

$$\varrho(\mathbf{A}) := \max\{|\lambda| \,|\, \lambda \in \mathbf{Z}(\mathbf{A})\}.$$

No Largest MODULUS!

In contrast to the H-spectral radius $\rho(\mathbf{A})$, the Z-spectral radius $\varrho(\mathbf{A})$ of $\mathbf{A}$ may not be itself a positive Z-eigenvalue of $\mathbf{A}$.

Example

$$a_{1112} = 30, \quad a_{1212} = 1, \quad a_{1222} = 1, \quad a_{2111} = 6,$$
$$a_{2112} = 13, \quad a_{2122} = 37, \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

We solve:

$$\begin{cases} 30x_1^2x_2 + x_1x_2^2 + x_2^3 = \lambda x_1, \\ 6x_1^3 + 13x_1^2x_2 + 37x_1x_2^2 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

It is easy to check: **A** is irreducible and there are three Z-eigenpairs:

$$\lambda_1 = \frac{63}{5},\ x_1 = (\pm\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}));$$
$$\lambda_2 = \frac{-64}{5},\ x_2(\pm\frac{\sqrt{5}}{5}, \mp\frac{2\sqrt{5}}{5}));$$
$$\lambda_3 = -15,\ x_3 = (\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2})).$$

$\varrho(\mathbf{A}) = |\lambda_3| = 15$, but 15 is not a Z-eigenvalue of $\mathbf{A}$!

Let $\mathbf{A} \in R_+^{[m,n]}$,

$$\Lambda(\mathbf{A}) = \{\lambda \geq 0 \,|\, \lambda \in \mathbf{Z}(\mathbf{A})\}$$

is called the nonnegative spectrum .

- $\Lambda(\mathbf{A}) \neq \emptyset$ is compact, but not necessarily a finite set.

Example

$$a_{1112} = a_{2122} = 2 \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

The Z-eigenvalue problem is to solve:

$$\begin{cases} 2x_1^2 x_2 = \lambda x_1, \\ 2x_1 x_2^2 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

$$(x_1, x_2) \in P \cap S^1 \begin{cases} x_1 x_2 = 0 \Rightarrow \lambda = 0, \\ 0 < 2x_1 x_2 \leq 1 \Rightarrow \lambda = 2x_1 x_2. \end{cases}$$

$\Lambda(\mathbf{A}) = [0, 1]$.

Special classes of nonnegative tensors
1. Weakly symmetric tensors

$\mathbf{A} \in \mathbb{R}^{[m,n]}$ is called weakly symmetric [CPZ] if the associated homogeneous polynomial

$$\mathbf{A}x^m = f_{\mathbf{A}}(x) := \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

satisfies $\nabla f_{\mathbf{A}}(x) = m\mathbf{A}x^{m-1}$.

∘ Symmetric tensor $\Rightarrow$ weakly symmetric, but the converse is not true.

∘ Eigenvalue /eigenvector of a weakly symmetric tensor $\mathbf{A}$, $\Leftrightarrow$ critical value/ point of the function $f_{\mathbf{A}}$.

∘ If $\mathbf{A}$ is weakly symmetric, then

$$f_{\mathbf{A}}(x) = \frac{1}{m}\langle \nabla f_{\mathbf{A}}(x), x\rangle = \langle \mathbf{A}x^{m-1}, x\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Let

$$\lambda^* = \max\{\lambda \in \Lambda(\mathbf{A})\}.$$

$$\bar{\lambda} := \max_{x \in S^{n-1}} f_{\mathbf{A}}(x) = \max_{x \in S^{n-1}} \mathbf{A}x^m.$$

Definition

([CPZZ]) Let $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ be irreducible. We define the following two functions for all $x \in P^n \setminus \{0\}$:

$$\nu_*(x) := \min_{1 \le i \le n} \frac{(\mathbf{A}x^{m-1})_i}{x_i} \quad \text{and} \quad \nu^*(x) := \max_{1 \le i \le n} \frac{(\mathbf{A}x^{m-1})_i}{x_i}.$$

and

$$\varrho_* := \sup_{x \in (P^n)^\circ \cap S^{n-1}} \nu_*(x) \quad \text{and} \quad \varrho^* := \inf_{x \in (P^n)^\circ \cap S^{n-1}} \nu^*(x).$$

Theorem

[CPZZ] Assume $\mathbf{A} \in \mathbb{R}_+^{[m,n]}$ *is weakly symmetric and irreducible. Then*

1. $\Lambda(\mathbf{A}) \subseteq [\varrho^*, \varrho_*]$.
2. $\varrho(\mathbf{A}) = \bar{\lambda} = \lambda^* = \varrho_*$.

2. Transition probability tensors

A tensor $\mathbf{P} = (p_{i_1 i_2 \ldots i_m}) \in \mathbb{R}_+^{[m,n]}$ satisfying

$$\sum_{i_1=1}^{n} p_{i_1,i_2,\cdots,i_m} = 1, \quad 1 \leq i_2, \ldots, i_m \leq n,$$

is called a transition probability tensor. One studies the Z_1 eigenvalue problem for $\mathbf{P}$.

$$\mathbf{P}x^{m-1} = \lambda x, \; x \in \Delta_n = \{x \in P^n \,|\, \Sigma_{i=1}^n x_i = 1\}.$$

- $\Rightarrow \; \lambda = 1.$
- Z_1 eigenvectors of $\mathbf{P}$ $\Leftrightarrow$ fixed points of $T : x \mapsto \mathbf{P}x^{m-1}$ on Δ_n.

Theorem

(x_0, λ_0) is a Z_1-eigenpair $\Leftrightarrow$ $\left(\frac{x_0}{\|x_0\|_2}, \frac{\lambda_0}{\|x_0\|_2^{m-2}}\right)$ *is a Z-eigenpair.*

We focus on studying the classes of transition probability tensors, whose positive Z-eigenvector is unique. Let

$$\delta_m = \min_{V\subset\{1,2,\cdots n\}} [\min_{i_2,\cdots i_m} \Sigma_{i\in V} p_{ii_2\cdots i_m} + \min_{i_2,\cdots i_m} \Sigma_{i\in V'} p_{ii_2\cdots i_m}].$$

Theorem

(Contraction)(W. Li and M. Ng [LM]) Let $\mathbf{P}$ *be a transition probability tensor. If*

$$\delta_m > \frac{m-2}{m-1}, \quad (*)$$

then the mapping $T : x \to \mathbf{P}x^{m-1}$ *on the simplex* Δ_n *is a contraction.*

Corollary

(Uniqueness and Convergence)[LN] The map $T : x \mapsto \mathbf{P}x^{m-1}$ *on* Δ_n *possesses one and only one fixed point.*
$\forall\, x_0 \in \Delta_n$, $T^k(x_0)$ *tends to the fixed point of* T *as* $k \to \infty$. *Moreover, the iteration linearly converges.*

An easy verification: Let ([LN]).

$$Osc(\mathbf{P}) = \max_{i,i_2,\cdots,i_m,j_2,\cdots,j_m \in \{1,2,\cdots,n\}} |p_{ii_2\cdots,i_m} - p_{ij_2,\cdots,j_m}|$$

be the oscillation of **P**:

$$Osc(\mathbf{P}) < \frac{2}{n(m-1)} \Rightarrow (*).$$

Denote the $n-1$ dimensional simplex.

$$\Delta'_n = \{x' = (x_1, \cdots, x_{n-1}) \in P^{n-1} \,|\, 0 \le \Sigma_{k=1}^{n-1} x_k \le 1\},$$

Rewrite the mapping T on Δ_n by a mapping $R = R_n : \Delta'_n \to \Delta'_n$ as follow:

$$R(x') = (\mathbf{P}x^{m-1}|_{x_n = 1 - \Sigma_{j=1}^{n-1} x_j})_1^{n-1}.$$

Let $S = (s_{ij})$ be a symmetric matrix: $s_{ij} = \frac{1}{2}(r_{ij} + r_{ji})$, where

$$r_{ij}(x') = \frac{\partial R(x')_i}{\partial x_j}.$$

Theorem

(monotone [CZ]) If $\max_{x' \in \overline{\Delta'}} \gamma(S(x')) < 1,$ *where* $\gamma(S)$ *denote the largest eigenvalue of* S*, then the nonnegative* Z_1*-eigenvector of* $\mathbf{P}$ *is unique.*

Theorem

[CZ] If $\mathbf{P} \in \mathbb{R}^{[m,n]}$ *is a transition probability tensor and* $\exists\, k$ *such that*

$$d_k := |p_{i_1 i_2 \cdots i_m} - p_{k, i_2 \cdots i_m}| < \frac{1}{(n-1)(m-1)}, \quad \forall\, i_1, i_2, \cdots, i_m \in \{1, 2, \cdots n\},$$

then $\mathbf{P}$ *has unique fixed point in* Δ_n*.*

It is worth comparing Theorem (Contraction) and Theorem (monotone):

$$\min_k \max_{i_1 i_2 \cdots i_m} |p_{i_1 i_2 \cdots i_m} - p_{k i_2 \cdots i_m}| < \frac{1}{(n-1)(m-1)},$$

which bounds the oscillations between all elements with the same last $m-1$ modes: $i_2, \cdots, i_m$. and

$$Osc(\mathbf{P}) = \max_{i, i_2 \cdots i_m, j_2 \cdots, j_m} |p_{i i_2 \cdots i_m} - p_{i j_2 \cdots j_m}| < \frac{2}{n(m-1)},$$

which bounds the oscillation with the same first mode: i .

Theorem

(Jacobian [CZ]) If $\mathbf{P}$ *is an irreducible transition probability tensor, and* T *is the mapping defined above. Assume* $\det(Id - J(x)) \neq 0$ *does not change sign on* $Fix(T)$ *. Then* T *has a unique fixed point.*

where

$$J(x) = \left(\frac{\partial(\mathbf{P}x^{m-1})_i}{\partial x_j}\right).$$

Questions

1. Find classes of nonnegative tensors, in which every tensor possesses unique positive Z eigenvalue with positive eigenvector.
2. Find classes of nonnegative tensors, in which the Z spectral radius is positively simple.
3. Find classes of nonnegative tensors, in which, the Z spectral radius is of largest modulus.
4. Find rapid algorithm for computing $\varrho(\mathbf{A})$.
5. If there are multiple positive $Z-$eigenvalues, how to compute all of them ?

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